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Lie Group Theory of the Bessel Equation of the First Kind of Integral Order

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We discuss the Bessel differential equation of the first kind of integral order and the associated functions from a Lie-group-theoretical background. All the familiar properties of the Bessel equation and functions are obtained. The analytic methodology developed in the study can easily be adapted to the study of some other special functions of mathematical physics.

INTRODUCTION

Recurrence relations and special functions of mathematical physics (Courant and Hilbert, 1953; Morse and Feshback; 1953; Rainville, 1960; Lebedev, 1965) have properties which, for the most part, are derived on the basis of the methods of classical analysis. An alternative to this mode of study of functions of mathematical physics is a group-theoretic approach (Vilenkin, 1968). This approach elucidates the geometric background of the special functions, such as rotations, translations, and the like. The group-theoretic approach to the derivation of the properties of the special functions simplifies considerably the complicated mathematical manipulations of power series and integral representations which characterize the study of the classical theory of the special functions. The following is the correspondence between the classical and group-theoretic approaches to the study of the special functions of mathematical physics:

(i) The addition theorem of the special functions becomes multiplication laws for the elements of the group of symmetry involved.

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(ii) The differential equations satisfied by special functions are obtained as limiting cases of the addition theorems, or as expressions of the fact that multiplication of group elements in the neighborhood of the identity (unit) element furnishes group elements whose properties are in close proximity to the parameters of the elements multiplied.

(iii) The integral relationships among classical special functions now derive from Frobenius' orthogonality relations for the matrix elements of irreducible representations as generalized for Lie groups by means of Hurwitz's invariance integers.

(iv) Lie groups can be considered as limiting cases of others, and this furnishes further relations between them.

For example, the Euclidean group of the plane can be obtained as a limit of the group of rotations in three-space, and so the elements of the representations of the former (Euclidean group of the plane) are limits of the representations of the latter group. While the former group relates to the bessel equation of the first kind, of integral order, and the associated Bessel functions, the latter group is related to the Jacobi functions. Clearly, elements of certain group representations are specified special functions of mathematical physics.

In this paper, we obtain the Bessel equation of the first kind of integral order n, and the associated Bessel (special) functions from the elements of the group representations of the Euclidean group E_2 for the plane. This approach, the Lie-group-theoretic approach, provides a good alternative to the conventional series method due to Frobenius. The technique can be extended to the study of other differential equations of mathematical physics and their associated special functions once applicable symmetry groups are found as well as their desired representations. A number of properties of such differential equations and the associated special functions can be obtained group-theoretically.

The rest of the paper is organized as follows:

In Section 1, the properties of the Euclidean group E_2 of the plane and Frobenius' method of induced representations are discussed. In Section 2, the complete representations of E_2 are applied to obtain the Bessel functions J_m of the first kind and of integral order, m. By the use of the general addition theorem, some well-known recurrence relations for the Bessel functions J_m are obtained.

In Section 3, a Helmholtz partial differential equation satisfied by each matrix element of the representation of the translation operator of E_2 is obtained. The Bessel differential equation of the first kind of integral order m, as well as its generating function, is obtained. In Section 4, we present concluding remarks.

1. PROPERTIES OF THE EUCLIDEAN GROUP E_2 OF THE PLANE AND THE FROBENIUS METHOD OF INDUCED REPRESENTATION OF E_2

The Euclidean group E_2 of the plane is the set of all transformations of the plane, of the form

$$T(\mathbf{a})R(\theta) \tag{1.1}$$

where $R(\theta)$ is a rotation of the plane about the origin by an angle θ, and $T(\mathbf{a})$ is a translation of the plane by the vector $\mathbf{a}$. The coordinates $(\mathbf{x}', \mathbf{y}')$ of an arbitrary point (x, y) following the transformation (1.1) are given by

$$(x', y') = T(\mathbf{a})R(\theta)\{(x, y)\} \tag{1.2}$$

i.e.,

$$\begin{aligned} x' &= x \cos \theta - y \sin \theta + a \\ y' &= x \sin \theta + y \cos \theta + b \end{aligned} \tag{1.3}$$

or

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \tag{1.4}$$

where a and b are the components of $\mathbf{a}$. The three parameters of E_2 are thus a, b, and θ, and the infinitesimal generators of the Lie group of the continuous group E_2 or of the corresponding Lie algebra are

$$L_a, \quad L_b, \quad L_\theta$$

These are calculated in the differential form as follows:

Replace θ, a, b by their infinitesimals $\delta\theta$, δa, and δb, respectively, to obtain infinitesimal rotations and translations, namely:

$$\begin{aligned} x' &= x \cos \delta\theta - y \sin \delta\theta + \delta a \\ y' &= x \sin \delta\theta + y \cos \delta\theta + \delta b \end{aligned} \tag{1.5}$$

i.e.,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x & -y\delta\theta \\ x\delta\theta & y \end{pmatrix} + \begin{pmatrix} \delta a \\ \delta b \end{pmatrix} \tag{1.6}$$

as $\delta\theta \to 0$. Equations (1.6) give

$$x' - x = \delta x = -y\delta\theta + \delta a$$

$$y' - y = \delta y = x\delta\theta + \delta b$$

so that

$$\frac{\partial x}{\partial \theta} = -y, \qquad \frac{\partial x}{\partial a} = 1, \qquad \frac{\partial x}{\partial b} = 0$$

$$\frac{\partial y}{\partial \theta} = x, \qquad \frac{\partial y}{\partial b} = 1, \qquad \frac{\partial y}{\partial a} = 0$$

and

$$L_\theta = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$L_a = \frac{\partial x}{\partial a}\frac{\partial}{\partial x} + \frac{\partial y}{\partial a}\frac{\partial}{\partial y} = \frac{\partial}{\partial x}$$

$$L_b = \frac{\partial x}{\partial b}\frac{\partial}{\partial x} + \frac{\partial y}{\partial b}\frac{\partial}{\partial y} = \frac{\partial}{\partial y} \tag{1.7}$$

which are the three infinitesimal generators of the Lie group of E_2, in the differential form.

A faithful matrix representation of the Euclidean group element can be obtained. This is done by associating with each point (x, y) in the plane a three-dimensional vector $(x, y, 1)$. Under the transformation $T(\mathbf{a})R(\theta)$, the point $(x, y, 1)$ becomes $(x', y', 1)$,

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{1.8}$$

The matrix of transformation denoted by $M(\theta, a, b)$ is

$$M(\theta, a, b) = \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix}$$

The Lie algebra corresponding to the Lie group of E_2 can be obtained by calculating the derivatives of $M(\theta, a, b)$ with respect to the three parameters θ, a, b around the identity. The infinitesimal matrix generators of the algebra are

$$L_a = \frac{\partial M}{\partial a}(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_b = \frac{\partial M}{\partial b}(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_\theta = \frac{\partial M}{\partial \theta}(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.9}$$

The commutation relations are

$$[L_a, L_b] = 0, \qquad [L_a, L_\theta] = -L_b$$

$$[L_b, L_\theta] = -L_a \tag{1.10}$$

One notes that the general group element has been represented as the product of elements from two of its subgroups, which are the rotation group and the translation group, with $R(\theta)$ and $T(\mathbf{a})$ as the operators. These do not commute. In the product operation $R(\theta)T(\mathbf{a})$, $T(\mathbf{a})$ translates the origin into the point with coordinates (a, b), while the second operation $R(\theta)$ rotates this point (a, b) to the point with coordinates

$$(a \cos \theta - b \sin \theta, a \sin \theta + b \cos \theta)$$

This is equivalent to the transformation

$$T(\tilde{\theta}\mathbf{a})R(\theta)$$

where $\tilde{\theta}\mathbf{a}$ denotes a vector $\mathbf{a}$ rotated by $\tilde{\theta}$. Thus,

$$R(\theta)T(\mathbf{a}) = T(\tilde{\theta}\mathbf{a})R(\theta) \tag{1.11}$$

While the subgroup of translations is invariant, that of rotations is not. The group, however, can be written as the product of a normal subgroup and a subgroup NH such that

$$H \cap N = \{e\}$$

In other words, the group is a semidirect product group. We also note that the product of two group elements $T(\mathbf{a})R(\theta)$ and $T(\mathbf{a}')R(\theta')$ is

$$T(\mathbf{a})R(\theta)T(\mathbf{a}')R(\theta') = T(\mathbf{a})T(\tilde{\theta}\mathbf{a}')R(\theta)R(\theta')$$

$$= T(\mathbf{a} + \tilde{\theta}\mathbf{a}')R(\theta + \theta') \tag{1.12}$$

And, if R and S' are arbitrary rotations, and $T(\mathbf{a})$ and $T(\mathbf{b})$ arbitrary translations, we have

$$RT(\mathbf{a}) = T(R\mathbf{a})R$$

$$T(\mathbf{a})RT(\mathbf{b}) = T(\mathbf{a} + R\mathbf{b})RS$$

where $R\mathbf{a}$ denotes the vector $\mathbf{a}$ rotated by R.

We now discuss Frobenius' method of induced representation (Frobenius, 1896–1899; Schur, 1905; Burnside, 1911). The method is amenable to E_2. It provides that the representation D of a group G with a subgroup N also provides a representation of the subgroup N of G. The representation of N may be reducible, however, even if that of G, which is D, is irreducible. This is the case, since a subgroup may be invariant under the operators $D(n)$, $n \in N$, but not under all the $D(g)$, $g \in G$.

We now apply Frobenius' method of induced representation to the $T(\mathbf{a})$ subgroup of E_2, which is normal, noting that this subgroup is also Abelian, so that the invariant subspace is one-dimensional. The irreducible representations of the translation subgroup $T(\mathbf{a})$ are of the form

$$\exp(i\mathbf{p}\cdot\mathbf{a})$$

where $\mathbf{a}$ is the translation vector and $\mathbf{p}$ is an arbitrary vector of the space that labels the representation. If $D(\mathbf{a}, \mathbf{R}) \equiv D(\mathbf{a}, \theta)$ is the total representation of E_2, then $D(\mathbf{a}, I)$ is reduced. It is expected that there are vectors ψ in the representation space H that satisfy

$$D(\mathbf{a}, I)\psi = e^{i\mathbf{p}\cdot\mathbf{a}}\psi \tag{1.13}$$

Consider a vector $f(\mathbf{p})$ selected from a vectorspace H_p. Then all the vectors ψ of H_p are transformed by $D(\mathbf{a}, I)$ according to the equation

$$D(\mathbf{a}, I)\psi = e^{i\mathbf{p}\cdot\mathbf{a}}\psi \tag{1.14}$$

so that for $f(\mathbf{p}) \in H_p$, we get

$$D[\mathbf{a}, I]f(\mathbf{p}) = e^{i\mathbf{p}\cdot\mathbf{a}}f(\mathbf{p}) \tag{1.15}$$

We assume that $|\mathbf{p}|$ is fixed, so that f is a function of the direction ϕ of $\mathbf{p}$ only, where $0 \le \phi < 2\pi$. The angle θ is the polar angle of $\mathbf{p}$. From $D(\mathbf{a}, I)f(\mathbf{p}) = e^{i\mathbf{p}\cdot\mathbf{a}}f(\mathbf{p})$, we see that $D(\mathbf{a}, I)$ is a local operator that multiplies the value of f at each point $\mathbf{p}$ by $\exp(i\mathbf{p}\cdot\mathbf{a})$. Now,

$$\begin{aligned} D(a, 0)f(\phi) &= e^{ip|\mathbf{a}|}\cos(\beta-\phi)f(\phi) \\ &= e^{ipr\cos(\beta-\phi)}f(\phi) \end{aligned} \tag{1.16}$$

with $(\beta-\phi)$ as the angle between $\mathbf{p}$ and $\mathbf{a}$, with $|\mathbf{a}| = r$ (Fig. 1), and $|\mathbf{p}| = p$. The angle β is that made by the line of action of $\mathbf{a}$ with an arbitrary vector $\mathbf{p}$ in H_p, which labels the representation. The number p is an arbitrary

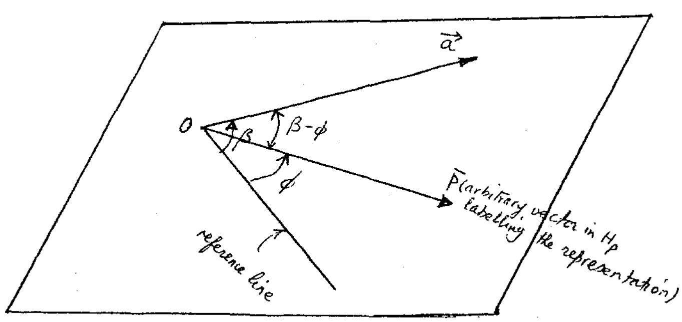

Fig. 1

positive number and is the index of the representation. Also,

$$D(0, \theta)f(\phi) = f(\phi - \theta)$$

2. COMPLETE MATRIX REPRESENTATION OF E_2

The complete representation of the element $T(\mathbf{a})R(\theta)$ of E_2, for $p \neq 0$, is given by

$$\begin{aligned} D(\mathbf{a}, \theta)f(\phi) &= [D(\mathbf{a}, 0)D(0, \theta)f]\phi \\ &= D(\mathbf{a}, 0)\{D(0, \theta)\}f(\phi) \\ &= \exp[ip|\mathbf{a}| \cos(\beta - \phi)]\, D(0, \theta)f(\phi) \\ &= \exp[ipr \cos(\beta - \phi + \theta)]\, f(\phi - \theta) \end{aligned} \tag{2.1}$$

There exists in a representation space a complete set of functions $f_{n\alpha}(\phi)$ satisfying the relation

$$D(0, \theta)f_{n\alpha}(\phi) = e^{-in\theta}f_{n\alpha}(\phi) \tag{2.2}$$

where the index α is there to indicate that there can be more than one function satisfying the above equation. If there is only one function of f, we get $D(0, \theta)f(\phi) = f(\phi - \theta)$.

Putting $\phi = 0$ gives

$$f_{n\alpha}(-\theta) = e^{-in\theta}f_{n\alpha}(0) \tag{2.3}$$

We can drop the index α, to get

$$f_n(-\theta) = e^{-in\theta}f_n(0)$$

Let $\theta \to -\theta$, and we obtain

$$f_n(\theta) = e^{in\theta}f_n(0)$$

We can replace θ by ϕ, to obtain

$$f_n(\phi) = e^{in\phi}f_n(0) \tag{2.4}$$

We choose as a normalization

$$f_n(0) = i^{-n}(2\pi)^{-1/2}$$

Now that the f_n are normalized, the representation they define is unitary. Then,

$$f_n(\phi) = \frac{i^{-n}}{(2\pi)^{1/2}}\, e^{in\phi} \tag{2.5}$$

The matrix elements of the translation generators can be readily calculated from the relation

$$D(\mathbf{a}, 0)f_n = \sum_m \Delta_p(\mathbf{a}, 0)_{mn} f_m \tag{2.6}$$

where the lhs is an operator $D(\mathbf{a}, 0)$ of G on f_n, and $\Delta_p(\mathbf{a}, 0)_{mn}$ is the matrix representing the operator $D(\mathbf{a}, 0)$, which is the translation operator of the translation subgroup of E_2.

Recall

$$D(\mathbf{a}, \theta)f_n(\phi) = \exp[irp \cos(\beta - \phi + \theta)] f_n(\phi - \theta) \tag{2.7}$$

and

$$D(\mathbf{a}, 0)f_n(\phi) = \exp[irp \cos(\beta - \phi)] f_n(\phi) \tag{2.8}$$

with

$$f_n(\phi) = \frac{1}{(2\pi)^{1/2}} i^{-n} e^{in\phi}$$

Hence,

$$\begin{aligned} D(\mathbf{a}, 0)f_n(\phi) &= \exp[irp \cos(\beta - \phi)] \frac{i^{-n}}{(2\pi)^{1/2}} e^{in\phi} \\ &= \frac{i^{-n} \exp[irp \cos(\beta - \phi)] e^{in\phi}}{(2\pi)^{1/2}} \end{aligned} \tag{2.9}$$

We can drop the $1/(2\pi)^{1/2}$ factor (no loss of generality), to obtain

$$D(\mathbf{a}, 0)f_n = \exp[irp \cos(\beta - \phi)] i^{-n} \exp(in\phi)$$

i.e.,

$$\begin{aligned} D(\mathbf{a}, 0)f_n &= \sum_m \Delta_p(\mathbf{a}, 0)_{mn} f_m \\ &= \exp[irp \cos(\beta - \phi)] i^{-n} \exp(in\phi) \end{aligned} \tag{2.10}$$

Use $f_n(\phi) = i^{-n} e^{in\phi}$ [on dropping the $1/(2\pi)^{1/2}$ factor], to obtain

$$f_m(\phi) = i^{-m} e^{im\phi}$$

so that

$$\sum_m \Delta(\mathbf{a}, 0)_{mn} f_m = \sum_m \Delta(\mathbf{a}, 0)_{mn} i^{-m} e^{im\phi}$$

i.e.,

$$\begin{aligned} i^{-n} \exp(in\phi) \exp[ipr \cos(\beta - \phi)] &= \sum_m \Delta(\mathbf{a}, 0)_{mn} f_m \\ &= \sum_m \Delta(\mathbf{a}, 0)_{mn} i^{-m} \exp(im\phi) \end{aligned} \tag{2.11}$$

This shows that $\Delta(\mathbf{a}, 0)_{mn}$ is the coefficient of $i^{-m} e^{im\phi}$ in the Fourier expansion of

$$i^{-n} \exp[ipr \cos(\beta - \phi)] \exp(in\phi)$$

In which case,

$$\Delta(\mathbf{a}, 0)_{mn} = \frac{i^{m-n}}{2\pi} \int_0^{2\pi} \exp[irp\cos(\beta - \phi)] \exp[i(n-m)\phi]\, d\phi \tag{2.12}$$

We integrate the rhs by changing the variable of integration:

$$\xi = \beta - \phi - \frac{\pi}{2}$$

so that $d\xi = -d\phi$, and so

$$\Delta_p(\mathbf{a}, 0)_{mn} = (-1)^{m-n} \exp[i(n-m)\beta]\, J_{m-n}(pr) \tag{2.13}$$

whereby one identifies

$$J_{m-n}(pr) = \frac{1}{2\pi} \int_0^{2\pi} \exp(-irp \sin \xi) \exp[i(m-n)\xi]\, d\xi \tag{2.14}$$

This is a familiar integral representation of $J_{m-n}(pr)$. It follows that

$$Jm(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(-ix \sin \xi) \exp(im\xi)\, d\xi \tag{2.15}$$

which is a familiar integral representation of the Bessel function of the first kind of integral order *m*.

We next obtain the complete matrix representation $\Delta p(\mathbf{a}, \theta)_{mn}$ of $D(\mathbf{a}, \theta)$. We recall that

$$\begin{aligned} D(0, \theta) f_n(\phi) = e^{-in\theta} f_n(\phi) &= e^{in\phi} i^{-n}\, e^{-in\theta} \\ &= e^{in(\phi-\theta)} i^{-n} \end{aligned}$$

Then

$$\begin{aligned} \sum_m \Delta(0, \theta)_{mn} f_n(\phi) &= \sum_m \Delta(0, \theta)_{mn}\, e^{-m}\, e^{im\phi} \\ &= i^{-n}\, e^{in(\phi-\theta)} \end{aligned} \tag{2.16}$$

showing that $\Delta(0, \theta)_{mn}$ is the coefficient of $e^{im\phi} i^{-m}$ in the Fourier expansion of $e^{in(\phi-\theta)}$, in which case,

$$\begin{aligned} \Delta(0, \theta)_{mn} &= \frac{i^{m-n}}{2\pi} \int_0^{2\pi} \exp[in(\phi - \theta)]\, d\phi \\ &= \frac{i^{m-n}}{2\pi} \exp(-in\theta) \end{aligned} \tag{2.17}$$

The complete representation $\Delta p(\mathbf{a}, \theta)_{mn}$ is now given, with the matrix elements, as

$$\begin{aligned}\Delta_p(\mathbf{a}, \theta)_{mn} &= [\Delta_p(\mathbf{a}, 0)\Delta_p(0, \theta)]_{mn} \\ &= (-1)^{m-n} \exp(-im\beta) \exp(in\beta)\, J_{m-n}(pr) \exp(-in\theta) \\ &= (-1)^{m-n} \exp(-im\beta)\, J_{m-n}(pr) \exp[in(\beta-\theta)] \end{aligned} \tag{2.18}$$

where (r, β) are the polar coordinates of $\mathbf{a}$, such that $|\mathbf{a}| = r$, $\arg(\mathbf{a}) = \beta$.

We now obtain the power series of a Bessel function of the first kind, of integral order, $J_m(a)$. Recall

$$\Delta_p(\mathbf{a}, \theta)_{mn} = (-1)^{m-n} \exp(-im\beta) J_{m-n}(pr)\, e^{in(\beta-\theta)}$$

The index p which labels the representation is irrelevant to the development of a power series for J_m and its other properties. We therefore take p as 1. Consider the transformations by the vector $\mathbf{a}$, parallel to the x axis of the plane, in order to obtain the power series for J_m. In this case the equation

$$D(\mathbf{a}, 0) f_n = \sum_m \Delta_p(\mathbf{a}, 0)_{mn} f_m$$

becomes

$$D(\mathbf{a}, 0)\psi_n = \sum_m (-1)^{m-n} \exp[i(n-m)\beta]\, J_{m-n}(pr)\psi_m \tag{2.19}$$

(evaluated at $p = 1$, as suggested). Here $|\mathbf{a}| = r = a$. The polar angle of $\mathbf{a}$ is β, and since $\mathbf{a}$ is parallel to the x axis of the plane, we have that $\beta = 0$. Hence,

$$D(\mathbf{a}, 0)\psi_n = \sum_m (-1)^{m-n} J_{m-n}(a)\psi_m \tag{2.20}$$

But a translation of a distance a in the x direction can be written operationally as

$$D(\mathbf{a}, 0) = e^{aL_a} \quad \text{(by exponentiation)} \tag{2.21}$$

where L_a is the corresponding generator of the Lie algebra corresponding to the subgroup of translation by the vector $\mathbf{a}$. Since the generators are not linearly independent, it is impossible to construct representations of the Euclidean group E_2 from the commutation relations:

$$[L_a, L_b] = 0, \qquad [L_a, L_\theta] = -L_b, \qquad [L_b, L_\theta] = L_a$$

The method of constructing irreducible representations for E_2 is to calculate, first, the irreducible representations of the Lie algebra. We need three independent generators other than L_a, L_b, L_θ. One is interested in unitary group representations. We try to find skew-Hermitian representations of the algebra. Consider the infinitesimal operators P^+, P^- defined by

$$P^+ = L_a + iL_b; \qquad P^- = (-P^+)^\dagger = L_a - iL_b \tag{2.22}$$

Recall the commutation relations of L_a, L_b, L_θ, namely

$$[L_\theta, L_a] = -[L_a, L_\theta] = -L_b$$

Now

$$[L_\theta, P^+] = -L_b - iL_a = -(L_b + iL_a)$$
$$= -i(L_a + iL_b) = -iP^+$$
$$[L_\theta, P^-] = iP^-$$

The Casimir invariant operator is

$$P^2 = L_a^2 + L_b^2 = P^+P^- = P^-P^+ \tag{2.23}$$

with P^2 commuting with L_a, L_b, and L_θ. Therefore P^2 is, if the representation is irreducible, a nonpositive real constant, which we denote by $-p^2$; i.e.,

$$P^2 = -p^2, \qquad p \neq 0 \tag{2.24}$$

From $L_a + iL_b = P^+$, $L_a - iL_b = P^-$, we obtain

$$L_a = \frac{1}{2}(P^+ + P^-), \qquad L_b = \frac{1}{2i}(P^+ - P^-)$$

Then

$$D(\mathbf{a}, 0) = \exp(aL_a) = \exp\left(a\frac{P^+ + P^-}{2}\right)$$
$$= \sum_{s=0}^{\infty} \frac{1}{s!}(P^+ + P^-)^s \left(\frac{a}{2}\right)^s$$
$$= \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{1}{s!} \frac{s!}{r!(s-r)!}(P^+)^r(P^-)^{s-r}\left(\frac{a}{2}\right)^s$$
$$= \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{1}{r!(s-r)!}\left(\frac{a}{2}\right)^s (P^+)^r(P^-)^{s-r} \tag{2.25}$$

It is necessary at this stage to invoke group properties in order to ensure that the representation of the algebra should correspond to a representation of E_2. The property to be invoked is that E_2 has a compact subgroup which is the rotation group. The chosen parametrization, in which $R(2\pi) = I$, requires that the eigenvalue of L_θ be *in*, where *n* is an integer, in order that $e^{2\pi L_\theta} = I$. We take ψ_n as a normalized eigenvector of L_θ satisfying

$$L_\theta\psi_n = -in\psi_n \tag{2.26}$$

There are two different cases: The first case is that for which $p^2 = 0$, implying

$$P^+P^-\psi_n = P^-P^+\psi_n = 0$$

so that

$$(\psi_n, P^+P^-\psi_n) = -|P^-\psi_n|^2 = 0$$

in which case

$$P^{-}\psi_n = 0$$

Similarly, $P^{+}\psi_n = 0$.

For this case the complete representation is defined by ψ_n which is one-dimensional and is of the rotation subgroup. This representation is of little interest.

The second case is that in which $p^2 > 0$. For all $\mathbf{u}$ in the domain of P^{+} and P^{-}, we have that $P^{+}\mathbf{u}$ and $P^{-}\mathbf{u}$ are nonzero, otherwise $p^2 = 0$. We consider $P^{+}\psi_n$. With this

$$[L_\theta, P^{+}] = -iP^{+}$$

gives

$$L_\theta(P^{+}\psi_n) = P^{+}L_\theta\psi_n - iP^{+}\psi_n$$

Using $L_\theta\psi_n = -in\psi_n$, this becomes

$$\begin{aligned} L_\theta(P^{+}\psi_n) &= P^{+}(-in\psi_n) - iP^{+}\psi_n \\ &= -i(n+1)P^{+}\psi_n \end{aligned}$$

indicating that $P^{+}\psi_n$ is an eigenvector of L_θ, corresponding to the eigenvalue $-i(n+1)$.

Similarly, $P^{-}\psi_n$ is an eigenvector of L_θ corresponding to the eigenvalue $-i(n-1)$. The nonnormalized eigenvectors $P^{+}\psi$, $P^{-}\psi_n$ satisfy

$$|P^{+}\psi_n|^2 = (\psi_n, P^{-}P^{+}\psi_n) = p^2$$

and

$$|P^{-}\psi_n|^2 = p^2$$

We next define the normalized eigenvectors of L_θ by

$$\psi_{n+1} = -\frac{P^{+}}{p}\psi_n, \qquad \psi_{n-1} = \frac{P^{-}}{p}\psi_n$$

The phases of ψ_{n+1}, ψ_{n-1} can be fixed arbitrarily; these have been chosen so that the representations obtained will agree with the equation of the complete representation

$$\Delta p(\mathbf{a}, \theta)_{mn} = (-1)^{m-n}\exp(-im\beta)J_{m-n}(pr)\exp[in(\beta-\theta)]$$

We inductively define

$$\psi_{n+m} = \left(\frac{-P^{+}}{p}\right)^m \psi_n, \qquad \psi_{n-m} = \left(\frac{P^{-}}{p}\right)^m \psi_n$$

which are again eigenvectors of L_θ corresponding to the eigenvalues

$-i(n+m)$ and $-i(n-m)$, respectively. We note that

$$\left(\frac{-P^+}{p}\right)\psi_{m+n} = \psi_{n+m+1}$$

$$\left(\frac{P^-}{p}\right)\psi_{n-m} = \psi_{n-m-1}$$

for $m > 0$, and

$$\frac{P^-}{p}\psi_{n+m} = \psi_{n+m-1}$$

$$\left(\frac{P^+}{p}\right)\psi_{n-m} = \psi_{n-m+1}$$

for $m \geq 1$. One concludes that the eigenvectors ψ_m, $m = n, n\pm 1$, provide a complete definition of the Lie algebra by

$$\begin{aligned} L_\theta\psi_m &= -im\psi_m \\ P^+\psi_m &= -p\psi_{m+1} \\ P^-\psi_m &= p\psi_{m-1}, \qquad m = n, n\pm 1 \end{aligned}$$

This construction is independent of the choice of the eigenvector ψ_n in

$$L_\theta\psi_n = -in\psi_n$$

for if another eigenvector of L_θ had been chosen, the same sequence of eigenvectors would have been found. The eigenvectors of L_θ are also nondegenerate. Degeneracy would lead to reducibility in representation. The representations are necessarily infinite-dimensional, since the eigenvectors $\psi_{n\pm m}$ defined by

$$\psi_{n+m} = \left(\frac{-P^+}{p}\right)^m \psi_n, \qquad \psi_{n-m} = \left(\frac{P^-}{p}\right)^m \psi_n$$

cannot vanish for any value of m and are necessarily linearly independent.

We now return to (2.25). Recall equation (2.6) and replace f_n by ψ_n, to obtain

$$D(\mathbf{a}, 0)\psi_n = \sum_m (-1)^{m-n} J_{m-n}(a)\psi_m$$

In the special case of $n = 0$, we get

$$D(\mathbf{a}, 0)\psi_0 = \sum_m (-1)^m J_m(a)\psi_m$$

Relate this to (2.25) to obtain

$$\begin{aligned} D(\mathbf{a}, 0) &= \sum_m (-1)^m J_m(a)\psi_m \\ &= \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{1}{r!(s-r)!}\left(\frac{a}{2}\right)^s (P^+)^r (P^-)^{s-r} \end{aligned}$$

We recall

$$\left(\frac{P^-}{p}\right)\psi_{n+m}=\psi_{n+m-1}; \qquad \left(\frac{-P^+}{p}\right)\psi_{m+n}=\psi_{n+m+1}$$

with $n=0$; i.e.,

$$\frac{P^-}{p}\psi_m=\psi_{m-1}; \qquad \frac{-P^+}{p}\psi_m=\psi_{m+1}$$

or,

$$P^-\psi_m=p\psi_{m-1}; \qquad P^+\psi_m=-p\psi_{m+1}$$

From these we obtain

$$(P^+)^r(P^-)^{s-r}\psi_0=(-1)^r\psi_{2r-s}$$

We substitute this, to obtain

$$\begin{aligned}\sum_m(-1)^m J_m(a)\psi_m &= \sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(s-r)!}\left(\frac{a}{2}\right)^s\psi_{2r-s}\\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^r}{r!s!}\left(\frac{a}{2}\right)^{r+s}\psi_{2r-(r+s)}\\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^r}{r!s!}\left(\frac{a}{2}\right)^{r+s}\psi_{r-s}\end{aligned}\tag{2.27}$$

The series for J_m is obtained by equating the coefficients of ψ_n on both sides of the identity. Two cases arise, namely

$$m\geq 0 \qquad \text{and} \qquad m<0$$

For $m\geq 0$, the terms on the rhs for which $r-s=m$ or $r=s+m$ give the desired result, which is the following:

$$J_m(a)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(s+m)!}\left(\frac{a}{2}\right)^{2s+m}\tag{2.28}$$

For $m<0$, the terms for which $s=r-m$ give the result

$$\begin{aligned}J_m(a)&=(-1)^m\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(r-m)!}\left(\frac{a}{2}\right)^{2r-m}\\ &=(-1)^m J_{-m}(a)\end{aligned}\tag{2.29}$$

implying that J_m and J_{-m}, for integral m, are linearly dependent and cannot be combined linearly to give a general solution of the Bessel differential equation of the first kind of order m. These power series are convergent for all values of the argument.

We next look at the general addition theorem and its implications for the Bessel function, which are recurrence relations. The most general addition theorem for E_2 is

$$\Delta(\mathbf{a}+\tilde{\theta}\mathbf{a}'+\theta')_{mn}=\sum\Delta(\mathbf{a},\theta)_{mp}\Delta(\mathbf{a}',\theta')_{pn}$$

We can, without any loss of generality, put $\theta=\theta'=0$. Again we assume **a** parallel to the x axis, since this can be achieved by a simultaneous rotation of **a** and **a**$'$. We conveniently express the addition theorem in Cartesian coordinates. We write $\exp(-i\beta')=\cos\beta'-i\sin\beta'$, with $\cos\beta'=a'/r'$, $\sin\beta'=b'/r'$, so that

$$\cos\beta'-i\sin\beta'=\frac{a'-ib'}{r'}$$

Now,

$$\Delta(\mathbf{a}',0)_{pn}=(-1)^{p-n}\left[\frac{a'-ib'}{r'}\right]^{p-n}J_{p-n}(r')$$

with $n=0$. Since **a** is parallel to the x axis, then $\beta=0$; $\mathbf{a}+\mathbf{a}'$ has components $r+a'$, b', with $|\mathbf{a}|=r$. We note that

$$\Delta(\mathbf{a},0)_{mp}=(-1)^{m-p}J_{m-p}(r)$$

In the special case of $m=0$ (no loss of generality), we obtain

$$\left(\frac{r+a'-ib')}{R}\right)^m J_m(R)=\sum_p\left(\frac{a'-ib'}{r'}\right)^p J_{m-p}(r)J_p(r') \tag{2.30}$$

with

$$R^2=(r+a')^2+b'^2;\qquad r'^2=a'^2+b'^2$$

and summation is over all integral n values of p. In polar coordinates, we parametrize:

$$\frac{r+a'}{R}=\cos B,\qquad \frac{b'}{R}=\sin B$$

By De Moivre's theorem we obtain

$$\left(\frac{r+a'-ib'}{R}\right)^m=(\cos B-i\sin B)^m=e^{-imB}$$

so that

$$\left(\frac{r+a'-ib'}{R}\right)^m J_m(R)=\sum_p\left(\frac{a'-ib'}{r'}\right)^p J_{m-p}(r)J_p(r') \tag{**}$$

becomes

$$e^{-imBJ_m(R)}=\sum_p\exp(-ip\beta')J_{m-p}(r)J_p(r')$$

with

$$R^2 = (r+a')^2 + b'^2 = r^2 + a'^2 + 2ra' + b'2$$
$$= r^2 + 2rr' \cos \beta' + a'^2 + b'^2$$
$$= r^2 + 2rr' \cos \beta' + r'^2$$

In the special case of $b' = 0$, corresponding to the product of translations along the x axis, equation (**) reduces to

$$J_m(a+a') = \sum_p J_{m-p}(a) J_p(a') \tag{2.31}$$

Similarly, if $a' = 0$, the identity is, on replacing b' with b,

$$\left(\frac{a-ib}{(a^2+b^2)^{1/2}}\right)^m J_m((a^2+b^2)^{1/2}) = \sum_p (-1)^p J_{m-p}(a) J_p(b) \tag{2.32}$$

Equations (2.31) and (2.32) lead to well-known recurrence relations for the Bessel functions, which we obtain as follows:
Differentiate equation (2.31) with respect to a' and evaluate at $a' = 0$, to obtain

$$J'_m(a) = J_{m-1}(a) - J_{m+1}(a) \tag{2.33}$$

Similarly, equation (2.32) is differentiated with respect to b and evaluated at $b = 0$, to obtain

$$\frac{2m}{a} J_m(a) = J_{m-1}(a) J_{m+1}(a) \tag{2.34}$$

3. PARTIAL DIFFERENTIAL EQUATION OF HELMHOLTZ FOR $\Delta(a) \equiv \Delta(r, \beta)$ FOR EUCLIDEAN GROUP E_2 FOR THE PLANE

Consider the identity

$$\Delta(\mathbf{a}, 0)\Delta(\mathbf{a}', 0) = \Delta(\mathbf{a}+\mathbf{a}', 0) \tag{3.1}$$

where

$$\Delta(\mathbf{a}, 0)_{mn} = (-1)^{m-n} e^{i(n-m)\beta} J_{m-n}(pr)$$

(evaluated at $\beta = 0, p = 1$)

$$= (-1)^{m-n} J_{m-n}(r) \tag{3.2}$$

and

$$\Delta(\mathbf{a}', 0)_{mn} = (-1)^{m-n} \left(\frac{a'-ib'}{r'}\right)^{m-n} J_{m-n}(r') \tag{3.3}$$

The Helmholtz equation arises from the above identity [equation (3.1)]. Differentiate equation (3.1) with respect to a' and b' and evaluate the results at $a'=0$ to obtain

$$\frac{\partial \Delta}{\partial a}(\mathbf{a}) = \Delta(\mathbf{a})L_a \tag{3.4}$$

$$\frac{\partial \Delta}{\partial b}(\mathbf{b}) = \Delta(\mathbf{a})L_b \tag{3.5}$$

Alternatively, from $D(\mathbf{a}, 0) = e^{aL_a}$ (exponentiation), so that $\Delta(\mathbf{a}, 0)_{mn} = e^{aL_a}$, we obtain

$$\frac{\partial \Delta}{\partial a} = L_a\, e^{aL_a} = L_a \Delta$$

Similarly,

$$\frac{\partial \Delta}{\partial b} = \Delta L_b$$

Differentiate again with respect to a and b, respectively, and apply the Casimir operator $L_a^2 + L_b^2 = -p^2 = -1$ (no loss of generality), as follows:

$$\frac{\partial^2 \Delta}{\partial a^2} = \frac{\partial}{\partial a}(\Delta(\mathbf{a}))L_a$$

$$\frac{\partial^2 \Delta}{\partial b^2} = \frac{\partial}{\partial a}(\Delta(\mathbf{a}))L_b$$

to obtain

$$\begin{aligned}\frac{\partial^2 \Delta}{\partial a^2} + \frac{\partial^2 \Delta}{\partial b^2} &= \Delta(\mathbf{a})L_a^2 + \Delta(\mathbf{a})L_b^2 \\ &= (L_a^2 + L_b^2)\Delta(\mathbf{a}) \\ &= -p^2\Delta(\mathbf{a}) = -\Delta(\mathbf{a})\end{aligned}$$

Hence,

$$\frac{\partial^2 \Delta}{\partial a^2}(\mathbf{a}) + \frac{\partial^2 \Delta}{\partial b^2}(\mathbf{a}) + \Delta(\mathbf{a}) = 0 \tag{3.6}$$

which is the two-dimensional Helmholtz equation satisfied by each matrix element of the representation of the translation operator $D(\mathbf{a}, 0)$, with the matrix element of the representation as $\Delta(a, 0)_{mn}$. In polar coordinates (r, β) for the translation vector $\mathbf{a}$, we obtain

$$\Delta_{rr} + \frac{1}{r}\Delta_r + \frac{1}{r^2}\Delta_{\beta\beta} + \Delta(r, \beta) = 0 \tag{3.7}$$

or in terms of the Laplace operator

$$\nabla^2\Delta(r, \beta) + \Delta(r, \beta) = 0 \tag{3.8}$$

From the Helmholtz differential equation we deduce, in a straightforward manner, the Bessel differential equation of the first kind of integral order m, $J_m(r)$, with $|\mathbf{a}| = r$: Recall

$$\nabla^2\Delta(r, \beta) + \Delta(r, \beta) = 0$$

and

$$\Delta_p(\mathbf{a}, 0)_{mn} = (-1)^{m-n} \exp[i(n-m)\beta]\, J_{m-n}(pr)$$

with $p = 1$, as specialized earlier. Putting $\beta = 0$, $|\mathbf{a}| = r$, we obtain

$$J''_m(r) + \frac{1}{r} J'_m(r) + \left(1 - \frac{m^2}{r^2}\right) J_m(r) = 0 \tag{3.9}$$

which is the Bessel differential equation of the first kind of order m in $J_m(r)$, with m an integer.

We can also easily deduce some recurrence relations by combining

$$2J'_m(a) = J_{m-1}(a) - J_{m+1}(a)$$

and

$$\frac{2m}{a} J_m(a) = J_{m-1}(a) + J_{m+1}(a)$$

to obtain

$$J_{m-1}(r) = J'_m(r) + \frac{m}{r} J_m(r) \tag{3.10}$$

with $|\mathbf{a}| = a = r$; and

$$J_{m+1}(r) = -J'_m(r) + \frac{m}{r} J_m(r) \tag{3.11}$$

Equations (3.10) and (3.11) are well-known recurrence relations for the Bessel functions of interest.

By adding equations (3.10) and (3.11), we obtain another recurrence relation:

$$J_{m-1}(r) + J_{m+1}(r) = \frac{2m}{r} J_m(r) \tag{3.12}$$

The two relations (3.10) and (3.11) provide the factorization of the Bessel equation.

Finally, we work out the generating function for the Bessel equation from the matrix representation $\Delta p(\mathbf{a}, 0)_{mn}$ of $D(\mathbf{a}, 0)$. We recall that $\Delta_p(\mathbf{a}, 0)_{mn}$ have been defined as the coefficients of $i^{-m} e^{im\phi}$ in the Fourier expansion of $\exp[ir\cos(\beta-\phi)]\, i^{-n}\exp(in\phi)$. In the special case of $\beta=\pi/2$, for which $\Delta(\mathbf{a},0)_{mn}=i^{m-n}J_{m-n}(r)$, this property gives

$$i^{-n}\, e^{ir\sin\phi}\, e^{in\phi}=\sum_m i^{m-n}J_{m-n}(r)\, i^{-m}\, e^{in\phi} \tag{3.13}$$

Substitute $z=e^{i\phi}$, so that

$$\sin\phi=\frac{z-z^{-1}}{2i},\qquad \cos\phi=\frac{z+z^{-1}}{2}$$

to get

$$i^{-n}\exp\left(ir\frac{z-z^{-1}}{2i}\right)\exp(in\phi)=\sum_m i^{m-n}J_{m-n}(r)\, i^{-m}\exp(im\phi)$$

i.e.,

$$i^{-n}\exp\left(ir\frac{z-z^{-1}}{2i}\right)\exp(in\phi)=i^{-n}\sum_m J_{m-n}(r)\, i^{-m}\exp(im\phi)$$

or

$$\exp\left(ir\frac{z-z^{-1}}{2i}\right)\exp(in\phi)=\sum_m J_{m-n}(r)\, i^{-m}\exp(im\phi)$$

Put $n=0$, a special case, to obtain

$$\exp\left(ir\frac{z-z^{-1}}{2i}\right)=\sum_m J_m(r)z^m \tag{3.14}$$

This result has been demonstrated for $|z|=1$. The series converges for other values of z and can be extended to these values by analytic continuation. The lhs of equation (3.14) is called the generating function of the Bessel function $J_m(r)$. The result of this section can be extended to complex values of the arguments of the Bessel functions and for complex transformations of vectors $\mathbf{a}$ and $\mathbf{a}'$.

4. SUMMARY

We have been able to obtain the Bessel differential equation of the first kind of integral order, the related Bessel functions, their generating function, and some recurrence relations from a Lie-group-theoretic approach, the group of transformations being the Euclidean group E_2 for the plane. In addition, we have obtained a two-dimensional Helmholtz differential equation which is satisfied by each matrix element of the representation of the translation operator of E_2. Some interesting information can be gained from this differential equation.

One notes that the Lie-group-theoretic approach to the study of the special functions of mathematical physics is a consistent alternative to the method of power series and integral representations which characterizes the study of the classical theory of these special functions. The group-theoretic approach elucidates the geometric background of the special functions such as rotations and translations.

The analytic methodology thus developed in the study of the Bessel equation of the first kind of integral order and the related special functions and formulas can easily be applied to the study of some other special functions. For example, one can study the Euclidean group E_3 of rotations and translations in three dimensions. Applying the method of Frobenius already outlined, or that of Miller (1964), to relate the translation operator to the group representations of the elements of the group of translations, one obtains the spherical Bessel functions, the Neumann functions, and the spherical Hankel functions as well as their properties, Lie-group-theoretically. Results obtained group-theoretically compare well with those obtained by a different method by Friedman and Russek (1954) and by Danos and Maximon (1965). The associated Laguerre polynomials and the various properties of these functions can be derived group-theoretically from the irreducible unitary representation property of the group that has an algebra defined by the quantum mechanical position and momentum operators. It will be shown that the representation matrix elements are eigenfunctions of the two-dimensional harmonic oscillator problem and are closely related to hydrogen atom radial wave functions. A set of partner functions for the group representation can also be constructed. These partner functions are eigenfunctions of the harmonic oscillator problem. The transformation properties of the partner functions under the group can give rise to certain properties of Hermite polynomials.

It is expected that representation coefficients of somewhat more complex Lie groups than those of the simple ones we have studied and discussed will play an important role in the development of new special functions of mathematical physics, which, for want of an appropriate terminology, can be styled as transcendental special functions. We shall discuss some of these possibilities in a subsequent paper.

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